ON THE REALIZABILITY OF A GRAPH AS THE GRUENBERG–KEGEL GRAPH OF A FINITE GROUP

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We use the term "group" while meaning "finite group" and the term "graph" while meaning "undirected graph without loops and multiple edges".

In the finite group theory many researchers are interested in various problems of the study of groups by their arithmetical properties. One of such problems is the problem of the study of a group by some properties of its Gruenberg–Kegel graph.

Let G be a group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e., the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this graph, the vertex set is $\pi(G)$ and different vertices p and q are adjacent if and only if $pq \in \omega(G)$.

We say that a graph Γ with $|\pi(G)|$ vertices *is realizable as the Gruenberg–Kegel graph of a group* G if there exists a marking the vertices of Γ by different primes from $\pi(G)$ such that the marked graph is equal to $\Gamma(G)$. A graph Γ *is realizable as the Gruenberg–Kegel graph of a group* if Γ is realizable as the Gruenberg–Kegel graph of an appropriate group G.

The following problem arises.

Problem. *Let Γ be a graph. Is Γ realizable as the Gruenberg–Kegel graph of a group*?

Of course, in general, the problem has negative solution. For example, Gruenberg–Kegel theorem and the description of connected components of the Gruenberg–Kegel graphs for all simple non-abelian groups [1,2] imply that the graph consisting of five pairwise non-adajcent vertices (5-coclique) is not realizable as the Gruenberg–Kegel graph of a group.

There are not many works devoted to this interesting problem.

In the paper [3] it was shown that a graph Γ is realizable as the Gruenberg–Kegel graph of a solvable group if and only if its complement is 3-colorable and triangle free.

In unpublished graduate work of I. N. Zharkov [4], who was a student of V. D. Mazurov, it was proved that a chain is realizable as the Gruenberg–Kegel graph of a group if and only if the length of this chain is at most 4.

In the paper [5] it was shown that any graph with at most five vertices besides of 5-coclique is realizable as the Gruenberg–Kegel graph of a group.

In this talk, we give a solution of the mentioned problem for all complete bipartite graphs $K_{m,n}$, where $K_{m,n}$ is the graph with $m+n$ vertices whose vertices can be divided into two disjoint subsets U and V such that $|U| = m$, $|V| = n$ and vertices are adjacent if and only if they belong to different subsets. We prove the following theorem.

Theorem. *Let Γ be a complete bipartite graph $K_{m,n}$, where $m \leq n$. Then the following statements hold*:

(1) *Γ is realizable as the Gruenberg–Kegel graph of a group if and only if $m + n \leq 6$ and $(m, n) \neq (3, 3)$*;

(2) *if $m + n \leq 6$ and $(m, n) \neq (3, 3), (1, 5)$ then there exist infinitely many sets T of primes such that Γ is realizable as the Gruenberg–Kegel graph of a group G and $T = \pi(G)$*;

(3) *if $(m, n) = (1, 5)$ and Γ is realizable as the Gruenberg–Kegel graph of a group G then $\pi(G) = \{2, 3, 7, 13, 19, 37\}$, $O_2(G) \neq 1$ and $G/O_2(G) \cong {}^2G_2(27)$.*

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